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THE NUMERICAL SOLUTION OF STRAIN WAVE PROPAGATION IN ELASTICAL HELICAL SPRING

NUMERIČNA REŠITEV PROPAGACIJE DEFORMACIJSKEGA VALA V ELASTIČNI SPIRALNI VZMETI

Sami Ayadi[1], Ezzeddine Hadj-Taïeb[1], Guy Pluvinage[2]

[1]Unité de Recherche : Mécanique des fluides appliquée et modélisation, ENIS, B.P. W, 3038 Sfax, Tunisie
[2]Laboratoire de Fiabilité Mécanique, ENIM, Ile du Saulcy, 57045 Metz, France
samya@netcourrier.com

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When a helical spring is subjected to a rather large impact loading, significant axial and rotational oscillations can occur in the spring. A mathematical formulation is presented to describe non linear dynamic response of impacted helical springs. The governing equations for such motion are two coupled non-linear, hyperbolic, partial differential equations of second order. The axial and rotational strains and velocities are considered as principal dependent variables. Since the governing equations are non-linear, the solution of the system of equations can be obtained only by some approximate numerical technique. When the strains are small, the equations of motion are rendered linear. The numerical technique employed in this paper is the method of characteristics for, both, linear and non-linear wave propagation problems. In order to resolve the non-linear problem of the dynamic response of helical spring, the non-1inear characteristics method is used. The compatibility equations are integrated along the characteristics and written in difference form. Thus, the unknown values of the axial strain, rotational strain, axial velocity and rotational velocity at any point of the spring, can be determined by resolving a system of four simultaneous equations. For this system, the values of the coefficients and the known variables are computed by interpolation and integration along non-linear characteristic lines. The procedure must be slightly modified when the end points of the spring are involved. At both ends, in order to determine the unknown variables values, use is made of only two characteristics. The numerical results are obtained for helical spring under axial impact. The dynamic responses are computed and plotted for some sections of the spring.

Key words: helical spring, dynamic response, strains, method of characteristics, non linear behaviour

Ko je spiralna vzmet sunkovito močno obremenjena, lahko nastanejo v njej osna in rotacijsko nihanje. Predstavljena je matematična rešitev za opis nelinearnega odgovora obremenjene vzmeti. Ta odgovor lahko predstavimo z dvema povezanima nelinearnima portalnima hiperboličnima diferencialnima enačbama druge stopnje. Kot glavni spremenljivki so upoštevane osne in rotacijske deformacije. Enačbe niso linearne, zato so rešljive le s približno numerično tehniko. Pri majhnih deformacijah so enačbe nihanja linearne. Uporabljena numerična tehnika je metoda karakteristik za oboje, linearno in nelinearno propagacijo valovanja. S ciljem, da se najde rešitev za nelinearni problem, dinamičnega odgovora vzmeti, je uporabljena linearna karakteristika. Enačbe kompatibilnosti so integrirane vzdolž karakteristik in napisane v obliki diferenc. Tako način je mogoče določiti neznane vrednosti za aksialno in rotacijsko deformacijo in hitrost v vsaki točki vzmeti z rešitvijo sistema iz simultanih enačb. Za ta sistem so vrednosti koeficientov in znanih spremenljivk izračunane z interpolacijo in integracijo vzdolž nekarakterističnih linij. Proceduro je potrebno modificirati pri končnih točkah vzmeti. Za rešitev za ti dve točki in za določitev vrednosti neznanih spremenljivk sta uporabljeni le dve spremenljivki. Numerični rezultati so določeni za spiralno vzmet pri osni obremenitvi. Dinamični odgovori so izračunani in grafično prikazani za nekatere prereze vzmeti.

Ključne besede: spiralna vzmet, dinamični odgovor, deformacije, metoda karakteristik, nelinearno vedenje

1 INTRODUCTION

The dynamic behaviour of helical springs is an important engineering problem. In practice, helical springs are commonly used as structural elements in many mechanical applications (suspension systems, motor valve springs,…). The primary functions of springs are to absorb energy, to apply a definite force or torque, to support moving masses or isolate vibration,…

To simplify the analysis, it is generally assumed that the material is elastic. The design of helical springs requires two stages, the static and dynamic. The analytical solution to the static equations of cylindrical helical springs subjected to large deflections was obtained by Love [1].

In many research papers, the dynamic response of elastic material springs is investigated using various models. When a helical spring is subjected to a rather large impact loading, significant torsional oscillations can occur in the spring. The equations of motion, governing this behaviour, are derived in an article by Phillips and Costello [2]. Stokes [3] conducted an analytical and experimental program to investigate the radial expansion of helical springs due to longitudinal impact. The significance of torsional oscillations on the radial expansion of helical springs is presented in the work of Costello [4]. In this work a linear theory was presented and the analytical solution, obtained by the Laplace transform, did indicate rather large radial expansion under impact. Sinha and Costello [5] used a finite difference technique and the method of linear characteristics to solve numerically the non-linear partial differential equations in the time domain.

Mottershead [6] developed special finite element for solving the differential equations. Yilderim [7] developed an efficient numerical method based on the stiffness transfer matrix for predicting the natural frequencies of cylindrical helical spring. Becker et al. [8] also used the matrix transfer method to produce the natural frequencies of helical springs. Dammak et al. [9] developed an efficient two nodes finite elements with six degrees of freedom per node to model the behaviour of helical spring.

In this paper we extend the work of Sinha and Costello [5] to investigate numerically the non-linear behaviour of impacted springs using non linear characteristics method and to compare the results with linear theory. The numerical results of linear theory are obtained by the method of characteristics and by the Lax Wendroff finite differences method (Ayadi and Hadj-Taïeb [10]).

2 MATHEMATICAL MODEL

The equations which describe non linear one-dimensional dynamic behaviour of helical springs can be adapted from the analytical model developed by Phillips and Costello [2]. Applying the theory of dimensional analysis and the momentum equations, to an element of spring between two sections x and x+dx (**Figure 1a**), submitted to axial force F and torque T, yields the following equations of spring motion:

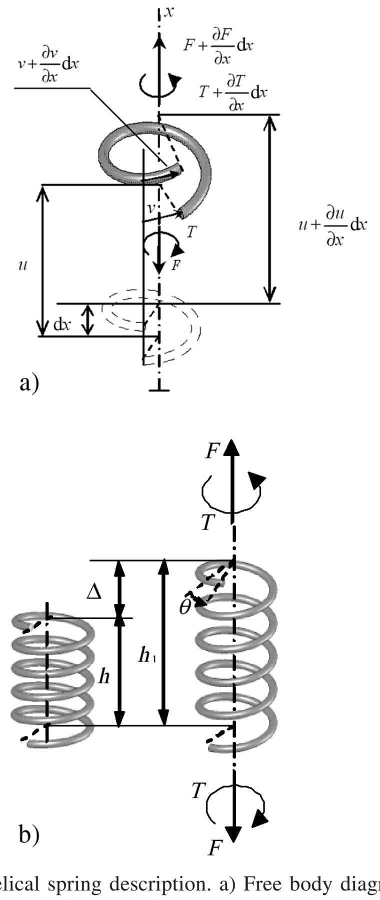

Figure 1: Helical spring description. a) Free body diagram of spring element, b) Static deflection of helical spring

Slika 1: Opis spiralne vzmeti; a) prostotelesni diagram elementa vzmeti; b) statični upogib spiralne vzmeti

$$a\frac{\partial^2 u}{\partial x^2}+b\frac{\partial^2 v}{\partial x^2}=e\frac{\partial^2 u}{\partial t^2} \tag{1}$$

$$b\frac{\partial^2 u}{\partial x^2}+c\frac{\partial^2 v}{\partial x^2}=e\frac{\partial^2 v}{\partial t^2} \tag{2}$$

where u is the axial displacement of the spring, $v = r\theta$ is the rotational displacement of the spring, r is the radius of the spring helix in the unstretched position, x is the axial co-ordinate along the spring and t is time.

The coefficients a, b, c and e, occurring in equations (1) and (2), are given by the expressions:

$$a=\frac{r^2}{EI}\cdot\frac{\partial F}{\partial \varepsilon}=(v_x\sin\alpha+\cos\alpha)(\sin\alpha)\cdot \left\{\frac{-\nu}{1+\nu}(v_x\sin\alpha+\cos\alpha)(\sin\alpha)+\frac{\cos^2\alpha}{\left[1-(1-u_x)^2\sin^2\alpha\right]^{3/2}}\right\} \tag{3}$$

$$b=\frac{r^2}{EI}\cdot\frac{\partial F}{\partial \beta}=\frac{r}{EI}\cdot\frac{\partial T}{\partial \varepsilon}=\sin^2\alpha\left\{\frac{(1+u_x)\cos^2\alpha}{\left[1-(1+u_x)^2\sin^2\alpha\right]^{1/2}}-\frac{\cos\alpha}{1+\nu}-\frac{2\nu}{1+\nu}(1+u_x)(v_x\sin\alpha+\cos\alpha)\right\} \tag{4}$$

$$c=\frac{r}{EI}\cdot\frac{\partial T}{\partial \varepsilon}=\sin\alpha\left[1-\frac{\nu}{1+\nu}(1+u_x)^2\ \sin^2\ \alpha\right] \tag{5}$$

$$e=\frac{Mr^2}{EIh} \tag{6}$$

where h is the length of the spring in the unstretched position, E is Young's modulus of the spring material, M is the total mass of the spring, I is the moment of inertia of the wire cross section, ν is Poisson's ratio of the spring material and α is the helix angle of the spring in the unstretched position. Thus, it is seen that the coefficients a, b, c and e are functions of $\varepsilon = u_x = \partial u/\partial x$ and $\beta = v_x = \partial v/\partial x$ and hence, the governing equations of motion are non-linear.

It can be seen from equations (3), (4) and (5) that when the strains are small, i.e., $|u_x| << 1$ and $|v_x| << 1$ the coefficients have the approximate values:

$$a=\left(1-\frac{\nu}{1+\nu}\cos^2\ \alpha\right)\sin\alpha$$

$$b=-\frac{\nu}{1+\nu}\sin^2\ \alpha\cos\alpha \tag{7}$$

$$c=\left(1-\frac{\nu}{1+\nu}\sin^2\ \alpha\right)\sin\alpha$$

If these values are employed in place of the actual non-constant coefficients, the equations of motion are rendered linear.

3 NUMERICAL SOLUTION

The numerical solution of the initial boundary value problem governed by the equations (1) and (2) may be obtained by the method of characteristics (Abbott [11], Chou and Mortimer [12], Hadj-Taïeb and Lili [13]). The method of characteristics, which is based on the propagation of the waves, is applied to obtain ordinary differential equations. In principle, it is not a numerical but an analytical solution method. However some of the necessary integrations are generally done numerically.

Equations (1) and (2) can be converted into a set of first order partial differential equations. Since $\partial u_x/\partial t = \partial u_t/\partial x$, $\partial v_x/\partial t = \partial v_t/\partial x$ and $(\partial u_x/\partial x)\mathrm{d}x + (\partial u_x/\partial t)\mathrm{d}t = \mathrm{d}u_x$ etc., the above set of equations (1) and (2), in matrix form, can be written as:

$$\begin{bmatrix} a & 0 & b & 0 & 0 & -1 & 0 & 0 \\ b & 0 & c & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ \mathrm{d}x & \mathrm{d}t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{d}x & \mathrm{d}t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{d}x & \mathrm{d}t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{d}x & \mathrm{d}t \end{bmatrix} \begin{bmatrix} \partial u_x/\partial x \\ \partial u_x/\partial t \\ \partial v_x/\partial x \\ \partial v_x/\partial t \\ \partial u_t/\partial x \\ \partial u_t/\partial t \\ \partial v_t/\partial x \\ \partial v_t/\partial t \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \mathrm{d}u_x \\ \mathrm{d}v_x \\ \mathrm{d}u_t \\ \mathrm{d}v_t \end{Bmatrix} \quad (8)$$

The characteristic directions are determined by setting the determinant of the coefficient matrix of equation (8) equal to zero. Hence the following equation results:

$$(ac-b^2)\left(\frac{dt}{dx}\right)^4 - (a+c)\left(\frac{dt}{dx}\right)^2 + 1 = 0 \quad (9)$$

The above equation has four roots which are:

$$\left(\frac{dt}{dx}\right)_{1,2} = +\left[\frac{(a+c) \mp \sqrt{(a-c)^2 + 4b^2}}{2(ac-b^2)}\right]^{1/2} \text{ and}$$
$$\left(\frac{dt}{dx}\right)_{3,4} = -\left[\frac{(a+c) \pm \sqrt{(a-c)^2 + 4b^2}}{2(ac-b^2)}\right]^{1/2} \quad (10)$$

When linear theory is used, we obtain:

$$\left(\frac{dx}{dt}\right)_{1,4} = \pm c_\mathrm{f} = \pm\sqrt{\frac{EIh}{Mr^2}\sin\alpha} = \pm\sqrt{\frac{\sin\alpha}{e}} \text{ and}$$
$$\left(\frac{dx}{dt}\right)_{2,3} = \pm c_\mathrm{s} = \pm\sqrt{\frac{EIh}{Mr^2}\frac{\sin\alpha}{1+\nu}} = \pm\sqrt{\frac{\sin\alpha}{e(1+\nu)}} \quad (11)$$

c_f is the fast speed of rotational waves (v_x, v_t) and c_s is the small speed of axial waves (u_x, u_t).

The four roots defined equations (10) or (11) are real and, hence, the system is hyperbolic. The canonical form of a hyperbolic system along the characteristics (sometimes called either 'Compatibility equations' or 'Riemann Invariant equations') can be determined by replacing any column of the coefficient matrix in equation (8) by the right-hand side column vector and setting the determinant equal to zero. The following equation results:

$$\left\{1 - c\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^2\right\}\mathrm{d}u_x + \left\{b\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^2\right\}\mathrm{d}v_x - $$
$$- \left\{1 - c\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^2\right\}\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)\mathrm{d}u_t + \left\{b\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^3\right\}\mathrm{d}v_t = 0 \quad (12)$$

In difference form, equation (12) becomes:

$$\left\{1 - c\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^2\right\}\Delta u_x + \left\{b\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^2\right\}\Delta v_x - $$
$$- \left\{1 - c\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^2\right\}\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)\Delta u_t + \left\{b\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^3\right\}\Delta v_t = 0 \quad (13)$$

Thus, the unknown values of (u_x, v_x, u_t and v_t), at any point L, as shown in **Figure 2**, can be determined by knowing their values at the points P, Q, R and S lying on the four characteristics passing through L and then solving four simultaneous equations obtained from equation (13). Although the characteristics are curved due to the non-linearity of equations (1) and (2), it will be assumed that LP, LQ, LR and LS are straight lines. Hence, equation (13) yields:

$$\left[1 - c\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^2_{1,\mathrm{P}}\right](u_{x\mathrm{L}} - u_{x\mathrm{P}}) + \left\{b\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^2_{1,\mathrm{P}}\right\}(v_{x\mathrm{L}} - v_{x\mathrm{P}}) -$$
$$- \left[1 - c\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^2_{1,\mathrm{P}}\right]\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)_{1,\mathrm{P}}(u_{t\mathrm{L}} - u_{t\mathrm{P}}) + \left\{b\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^3_{1,\mathrm{P}}\right\}(v_{t\mathrm{L}} - v_{t\mathrm{P}}) = 0$$

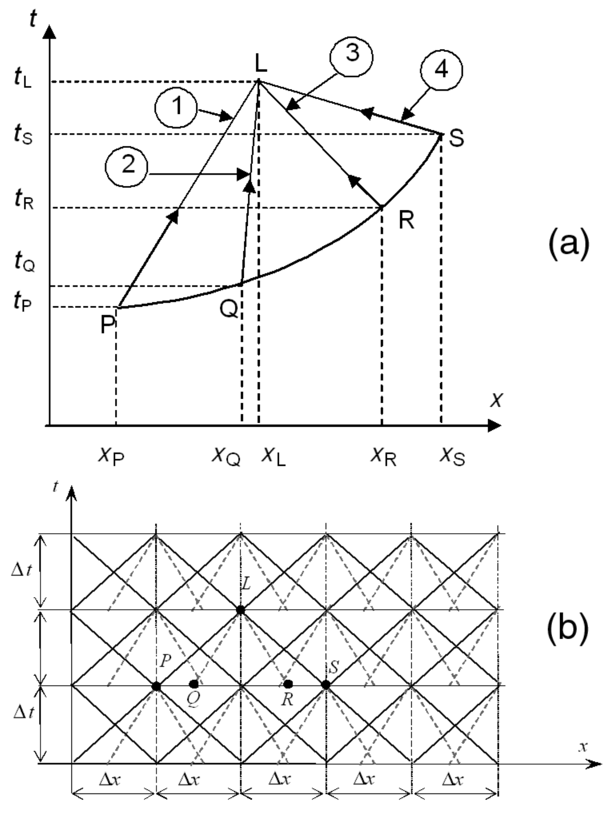

Figure 2: Method of characteristics. a) Non-linear theory, b) Linear theory

Slika 2: Metoda karakteristik; a) nelinearna teorija; b) linearna teorija where $u_{x\mathrm{L}}$, $v_{x\mathrm{L}}$, $u_{t\mathrm{L}}$ and $v_{t\mathrm{L}}$ are the unknown values at the point L; $u_{x\mathrm{P}}$, $v_{x\mathrm{P}}$, $u_{t\mathrm{P}}$ and $v_{t\mathrm{P}}$ are the known values at the point P; and $(\mathrm{d}t/\mathrm{d}u_x)_{1,\mathrm{P}}$ is the slope of the characteristics of family 1 passing through P. Three similar equations can be written for the points Q, R and S. By solving the four simultaneous equations obtained from equation (13), the values of $u_{x\mathrm{L}}$, $v_{x\mathrm{L}}$, $u_{t\mathrm{L}}$ and $v_{t\mathrm{L}}$ can be obtained at any point L. It should be noted that the values at the points P, Q, R and S, are computed by non-linear interpolation.

Figure 2b shows the characteristics in the case of linear theory where the wavespeeds c_f and c_s are constant.

4 NUMERICAL RESULTS FOR IMPACTED SPRING

Consider the hypothetical spring system shown by **Figure 3**. The parameters of the spring are: the original length of spring h = 48.26 cm, the helix angle α = 0.141815 rd, the radius of the spring r = 17.932 cm, the number of coils n = 3, the Poisson's ratio ν = 0.29, the wire radius r_f = 1.509 cm, the Young's modulus E = $20.685\cdot10^6$ N/cm^2, the initial compression Δ = 16.51 cm and the mass of the spring M = 19.146 kg.

Initial conditions

The initial conditions are:

$$u_x(x,0) = -\Delta/\mathrm{h} \quad \text{and} \quad v_x(x,0) = 0 \qquad (14)$$

$$u_t(x,0) = 0 \quad \text{and} \quad v_t(x,0) = 0 \qquad (15)$$

Boundary conditions

The dynamic response studied here is due to a given velocity at the impacted end of the spring x = 0 (see **Figure 4**). The boundary conditions are:

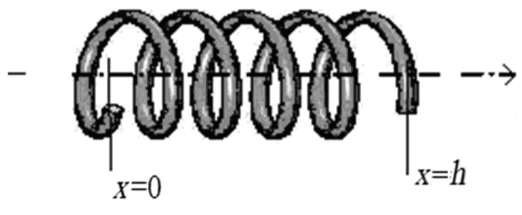

Figure 3: Helical spring boundaries
Slika 3: Meja spiralne vzmeti

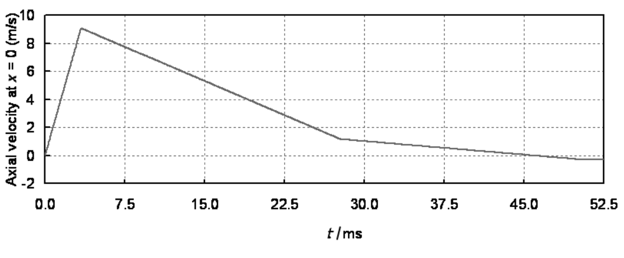

Figure 4: Axial velocity at the impacted end of the spring (x = 0)
Slika 4: Osna hitrost na sunkovito obremenjenem koncu vzmeti (x = 0)

$$u_\mathrm{t}(0,t) = \phi_1(t),\ v_\mathrm{t}(0,t) = 0,\ u_\mathrm{t}(h,t) = 0,\ v_\mathrm{t}(h,t) = 0 \qquad (16)$$

$\phi_1(t)$ is defined by the values given in **Table 1**.

Table 1: Axial velocity at x = 0

Time, t/ms	0	3.375	27.75	50.625
Axial velocity, /(m/s)	0	9.062	1.165	-0.3

The spring is divided into equidistant sections in the x direction: $\Delta x = h/N$. Two separate FORTRAN programs were run on a PC computer. The problem has been solved by the method of characteristics using N = 180 grid points for both linear and non-linear theories. In the case of linear theory the same problem has also been solved by the finite difference Lax-Wendroff method using N = 1000 grid points [Ayadi and Hadj-Taïeb (2006)].

The computed results by the method of characteristics for the linear and non-linear theories are shown in **Figures 5 and 6**. The axial and rotational strains are at the impacted end (x = 0).

As pointed by Phillips and Costello, the results of the plots show the necessity of solving the non-linear equations of motion for the spring under this type of loading. The linear theory is adequate for predicting the axial force in the spring but can lead to erroneous results in predicting the axial twisting moment and radial expansion of the spring (see **Table 2**).

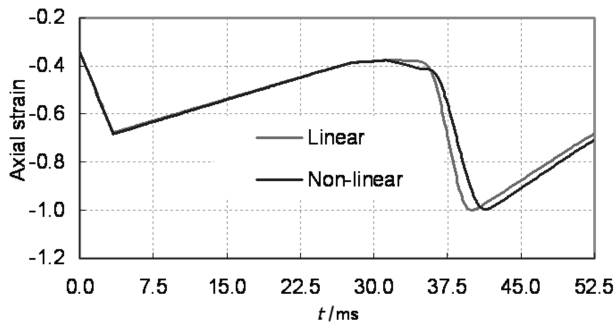

Figure 5: Axial strain at the impacted end (x = 0).
Slika 5: Osna deformacija na sunkovito obremenjenem koncu (x = 0)

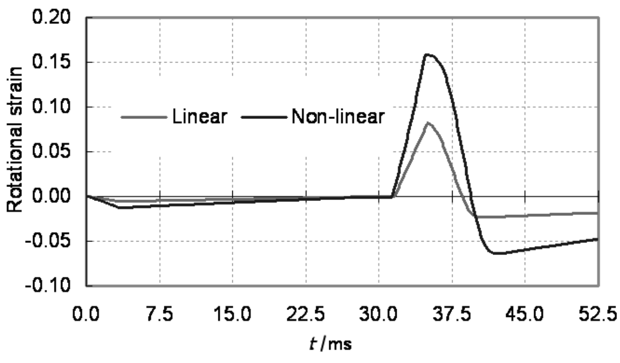

Figure 6: Rotational strain at the impacted end (x =0).
Slika 6: Rotacijska deformacija na sunkovito obremenjenem koncu (x = 0)

Table 2: Axial force, axial moment and radial expansion at $x = 0$ and $t = 0.0346$ sec

Time t/(ms)	Axial force (N)	Axial moment (mN)	Radial expansion (mm)
Linear theory	-11100	540.5	-0.950
Non linear theory	-11167.1	1155.4	-2.8194

It should be pointed out that once the axial and rotational strains are known, the stresses can be computed from the elementary strength of material formula. Generally, the most significant stresses occurring in a helical spring are due to the torsional moment acting on the wire cross section. Since the torsional moment on a cross section is due mainly to the

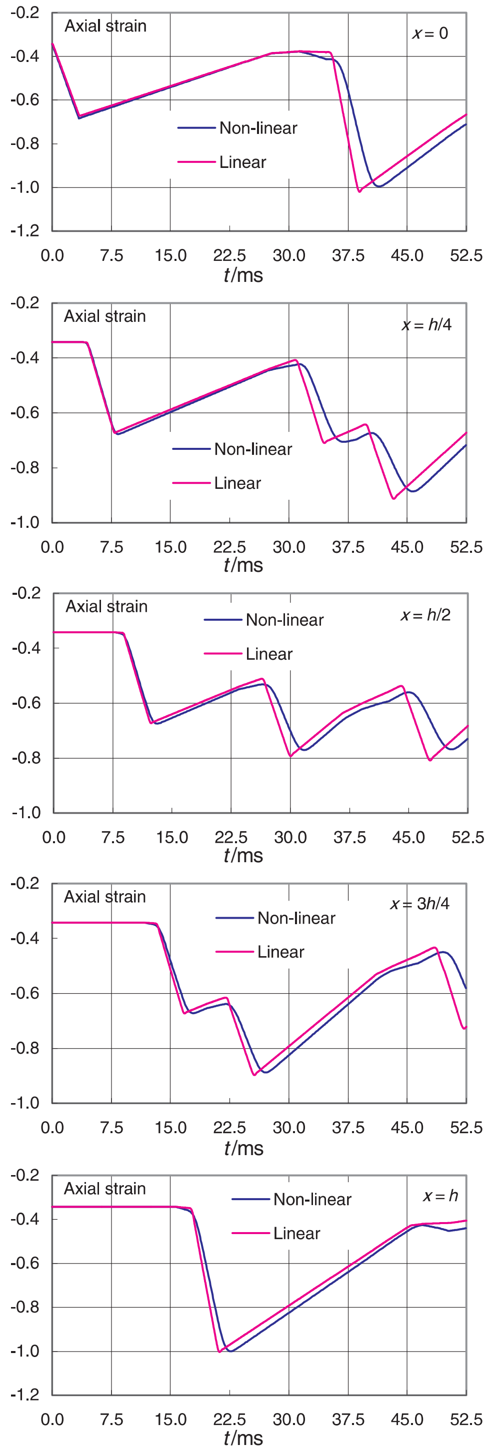

Figure 7: Axial strains in the spring
Slika 7: Osne deformacije v vzmeti

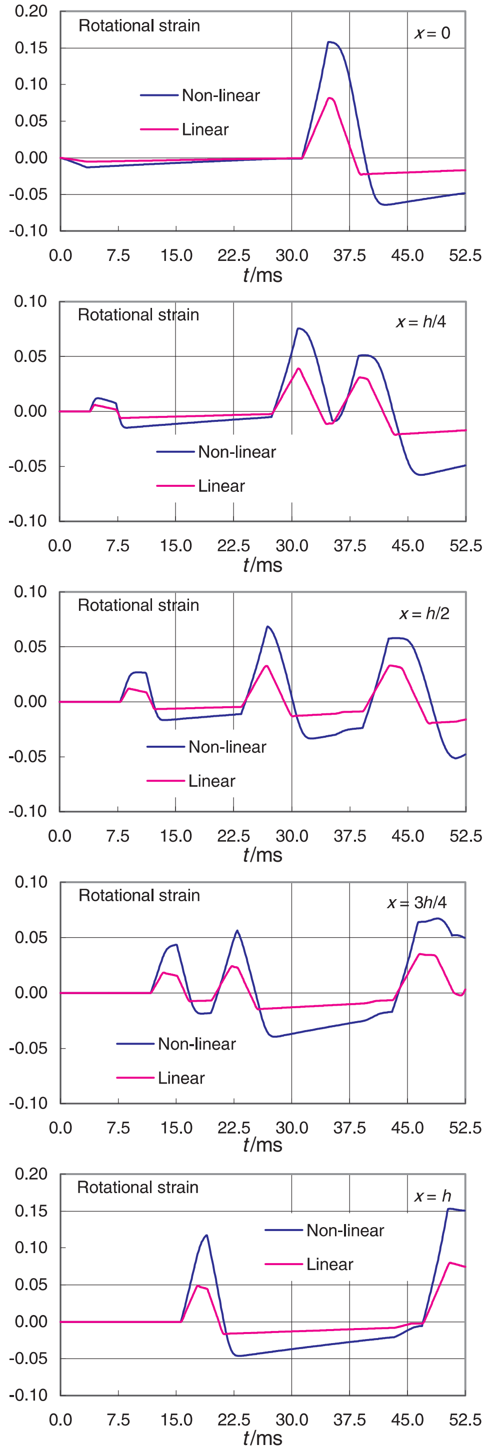

Figure 8: Rotational strains in the spring
Slika 8: Rotacijske deformacije v vzmeti

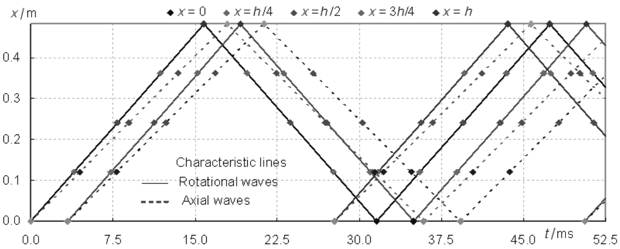

Figure 9: Characteristic lines for helical spring linear response dynamic

Slika 9: Karakteristične linije za linearni dinamični odgovor spiralne vzmeti

axial force in the spring, the linear theory is felt to be quite adequate for ca1culating the stresses in this example.

Figures 7 and 8 show the computed strain curves at some sections of the spring ($x = 0$, $x = h/4$, $x = h/2$, $x = 3h/4$ and $x = h$). It illustrates the phenomenon we are dealing with in the case of linear and non-linear spring dynamic responses. Due to the non-linearity of equations (1) and (2), the wave speeds are not constant and the characteristics lines are curved. Hence, the strain wave fronts are smoothly running. The computed strain results of the linear equations of motion presented in **Figures 7 and 8** are obtained by finite difference Lax-Wendroff scheme.

Figure 9 shows the characteristics lines for linear theory. At time $t = 0$, the spring is impacted and two waves, fast rotational strain wave and slow axial strain wave, travel the spring until they reach the other end x = h. The behaviour of characteristic paths of rotational wave differs from those of the axial one. The strain evolution would result from the velocity function applied at the impacted spring end, x = 0, and from the wave reflections at the two ends of the spring. It should be noted that the axial strain wave has an effect on the rotational strain. As it can be seen from the curves of **Figures 7 and 8**, the reflected rotational strain wave travelling from the end of the spring causes rotational strain to rise. But the reflected axial wave attenuates and limits the values of rotational strain. The process is repeated and indicated the influence of axial strain wave on the behaviour of rotational strain.

5 CONCLUSION

The numerical solution of the spring dynamic response has been presented in this paper. The solution is obtained with coupled two non-linear partial differential equations of the hyperbolic type. The two numerical methods employed are the method of curved characteristics and the finite-difference conservative method of Lax-Wendroff. The non-linear characteristics method requires the use of non-linear interpolation method to compute the strains evolution at any interior section of the spring.

The finite difference method is more practical and simulates correctly the strain waves propagation when the linear equations of motion are considered. Computed results obtained by this method agree favourably well with the numerical results based upon the characteristics method. The developed program has been applied to the large deformation analysis of helical springs under axial loading. It can be seen from the calculated results that the linear theory is reasonably accurate, as far as, the axial strain is concerned but is in considerable error for investigating the rotational strain.

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